ON LUSTERNIK-SCHNIRELMANN CATEGORY OF SO(10)

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Abstract. Let G be a compact connected Lie group and $p : E \to \Sigma A (A = \Sigma A_0)$ a principal G-bundle with a characteristic map $\alpha : A \to G$. We assume that there is a cone-decomposition $\{K_i \to F_{i-1} \to F_i \mid 1 \leq i \leq n,\ F_0 = \{*\}$ and $F_n \simeq X\}$ of G of length m. Our main theorem is as follows: we have $\mathrm{cat}(X) \leq m+1$, if the characteristic map α is compressible into F_1 and the Berstein-Hilton Hopf invariant $H_1(\alpha) = 0 \in [A, \Omega F_1 * \Omega F_1]$. We also apply it to the principal bundle $\mathrm{SO}(9) \hookrightarrow \mathrm{SO}(10) \to S^9$ to determine the L-S category of SO(10).

1. Introduction

In this paper, we work in the category of pointed CW-complex and don't distinguish a map from its homotopy class to make the arguments simpler. The Lusternik-Schnirelmann category of a space X is the least integer n such that there exists an open covering $U_0, \ldots U_n$ of X with each U_i contractible in the space X. We denote this by $\mathrm{cat}(X) = n$ and if no such integer exists, we write $\mathrm{cat}(X) = \infty$.

Theorem 1.1 (Ganea [3]). *Let X be a connected space. Then there is a sequence of fibrations $F_nX \to G_nX \to X$, natural with respect to X so that $\mathrm{cat}(X) \leq n$ if and only if the fibration $G_nX \to X$ has a cross-section.*

Here, F_nX has the homotopy type of $E^{n+1}\Omega X = \Omega X^{*(n+1)}$ the $(n+1)$-fold join of ΩX and G_nX has the homotopy type of the ΩX-projective n-space $P^n\Omega X$ in the sense of Stasheff [12] equipped with the composition $e^X_n : P^n\Omega X \hookrightarrow P^\infty\Omega X \simeq X$, where e^X_1 is given by the evaluation map (see also [4]).

Let R be a commutative ring and X a connected space. The cup-length of X with coefficients in R is the least non-negative integer k (or ∞) such that all $(k+1)$-fold cup products vanish in the reduced cohomology $\tilde{H}^*(X; R)$. We denote this integer k by $\mathrm{cup}(X; R)$ following Iwase [6].

In 1967, Ganea introduced in [3] a homotopy invariant $\mathrm{Cat}(X)$ for a space X, modifying Fox's strong category. In the same paper, he gave the following characterization using the notion of a cone-decomposition.

Definition 1.2 (Ganea [3]). The strong category $\mathrm{Cat}(X)$ of a connected space X is 0 if X is contractible and, otherwise, is equal to the least positive integer n

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such that there are cofibration sequences (called a cone-decomposition of length m)

$$\{K_i \to F_{i-1} \to F_i \mid 1 \leq i \leq n,\ F_0 = \{*\} \text{ and } F_n \simeq X\},$$

which is often called the cone-length of X.

The following inequalities among these invariants are well-known

$$\mathrm{cup}(X; R) \leq \mathrm{cat}(X) \leq \mathrm{Cat}(X).$$

Let $f : \Sigma X \to \Sigma Y$ be a map. We denote $H_1(f) \in [\Sigma X, \Omega\Sigma Y * \Omega\Sigma Y]$ by the Berstein-Hilton Hopf invariant (see Berstein and Hilton [1]).

The purpose of this paper is to prove the following theorem. Let G be a connected compact Lie group with a cone-decomposition of length m, that is, there are cofibration sequences

$$\{K_i \to F_{i-1} \to F_i \mid 1 \leq i \leq m\}$$

with $F_0 = *$ and $F_m \simeq G$. Let $G \hookrightarrow E \to \Sigma A (A = \Sigma A_0)$ be a principal bundle with a characteristic map $\alpha : A \to G$. The following is our main result.

Theorem 1.3. *If α is compressible into F_1, $H_1(\alpha) = 0 \in [A, \Omega F_1 * \Omega F_1]$ and K_m is a sphere, then we obtain* $\mathrm{cat}(E) \leq m + 1$.

In some application, we need to weaken the hypothesis slightly: suppose that there exists a space $F_1' = \Sigma K_1'$ with $K_1' \subset K_1$. Under the condition, the above theorem is extended as the following form.

Theorem 1.4. *If α is compressible into F_1', $H_1(\alpha) = 0$ and K_m is a sphere, then we obtain* $\mathrm{cat}(E) \leq m + 1$.

This yields, we obtain the following result.

Theorem 6.1. $\mathrm{cat}(\mathrm{SO}(10)) = 21$.

In Section 2 and 3, we construct a structure map and a cone-decomposition of some spaces which play the vital role in the proof of the main theorems. In Section 4, we show the important relation between a structure map and a cone-decomposition which are constructed in Section 2 and 3. In Section 5, we prove Theorem 1.4. In Section 6, we show some applications of Theorem 1.4.

2. Structure map associated with a filtration

Definition 2.1. The filtered space X is the space X equipped with a sequence of subspaces,

$$X \supset \cdots \supset X_n \supset X_{n-1} \supset \cdots \supset \{*\}.$$

We denote $i^X_{m,n} : X_m \to X_n$ by the inclusion map for $m < n$.

Definition 2.2. Suppose that the space X and Y are filtered by $\{X_n\}$ and $\{Y_n\}$, respectively. A filtered map $f : X \to Y$ is a filtration-preserving map, that is, $f(X_n) \subset Y_n$ for all n.

We denote $p^{\Omega X}_m$ by the map $E^m\Omega X \to P^{m-1}\Omega X$ in Theorem 1.1 and $\iota^{\Omega X}_{m,n}$: $P^m\Omega X \to P^n\Omega X$ by the inclusion map for $m < n$.

Proposition 2.3. *Let X and Y be filtered by $\{X_n\}$ and $\{Y_n\}$, respectively and a map $f : X \to Y$ be a filtered map. If the filtration of X is a cone-decomposition of X, say $\{L_i \xrightarrow{h_i} X_{i-1} \xrightarrow{i^X_{i-1,i}} X_i \,|\, 1 \le i \le n\}$, then there exist families of maps $\{f_i : X_i \to P^i\Omega Y_i \,|\, 0 \le i \le n\}$ and $\{g_i : L_i \to E^i\Omega Y_i \,|\, 1 \le i \le m\}$ such that $\{f_i\}$ and $\{g_i\}$ satisfy the following conditions.*

(1) *The following diagram is commutative.*

$$\begin{tikzcd}
L_i \arrow[r, "h_i"] \arrow[dd, "g_i"] & X_{i-1} \arrow[r, "i^X_{i-1,i}"] \arrow[d, "f_{i-1}"] & X_i \arrow[dd, "f_i"] \\
 & P^{i-1}\Omega Y_{i-1} \arrow[d, "P^{i-1}\Omega i^Y_{i-1,i}"] & \\
E^i\Omega Y_i \arrow[r, "P_i^{\Omega Y_i}"] & P^{i-1}\Omega Y_i \arrow[r, "\iota^{\Omega Y_i}_{i-1,i}"] & P^i\Omega Y_i.
\end{tikzcd}$$

(2) $e_i^{Y_i} \circ f_i = f|_{X_i}$

Proof. We prove the proposition by induction on i. In the case of $i = 1$, we put $g_1 = \mathrm{ad}(f|_{X_1})$, $f_0 = *$, and $f_1 = \Sigma\,\mathrm{ad}(f|_{X_1})$, respectively. Then the following diagram commutes.

$$\begin{tikzcd}
L_1 \arrow[r] \arrow[d, "g_1"] & * \arrow[r] \arrow[d, "f_0"] & \Sigma L_1 \arrow[d, "f_1"] \\
\Omega Y_1 \arrow[r] & * \arrow[r] & \Sigma\Omega Y_1.
\end{tikzcd}$$

Therefore, the condition (1) is satisfied when $i = 1$. Also the condition (2) holds from the following equation. For $t \wedge x \in \Sigma L_1$,

$$\begin{aligned}
e_1^{Y_1} \circ f_1(t \wedge x) &= \mathrm{ev} \circ \Sigma\,\mathrm{ad}(f|_{X_1})(t \wedge x) \\
&= \mathrm{ev}(t \wedge \Sigma\,\mathrm{ad}(f|_{X_1})(x)) \\
&= \mathrm{ad}(f|_{X_1})(x)(t) \\
&= (f|_{X_1})(t \wedge x).
\end{aligned}$$

Suppose (1) and (2) hold when $i = k-1$. First, we construct $g_k : L_k \to E^k\Omega Y_k$ from the exact sequence:

$$[L_k, E^k\Omega Y_k] \xrightarrow{p_k^{\Omega Y_k}{}_*} [L_k, P^{k-1}\Omega Y_k] \xrightarrow{e^{Y_k}_{k-1*}} [L_k, Y_k].$$

We use the equation

$$\begin{aligned}
e^{Y_k}_{k-1} \circ P^{k-1}\Omega i^Y_{k-1,k} \circ f_{k-1} &= i^Y_{k-1,k} \circ e^{Y_{k-1}}_{k-1} \circ f_{k-1} \\
&= i^Y_{k-1,k} \circ f|_{X_k} \circ i^X_{k-1,k}
\end{aligned}$$

and $i^X_{k-1,k} \circ h_{k-1} = 0$ by $L_k \xrightarrow{h_{k-1}} X_{k-1} \xrightarrow{i^X_{k-1,k}} X_k$ is the cofibre sequence. So, we have $e^{Y_k}_{k-1*}(P^{k-1}\Omega i^Y_{k-1,k} \circ f_{k-1} \circ h_{k-1}) = 0 \in [L_k, Y_k]$ and there exists a map

$g_k : L_k \to E^k\Omega Y_k$ such that ${p_k^{\Omega Y_k}}_*(g_k) = P^{k-1}\Omega i^Y_{k-1,k} \circ f_{k-1} \circ h_{k-1}$. Second, we construct a map $f_k : X_k \to P^k\Omega Y_k$. We define $f'_k : X_k \to P^k\Omega Y_k$ as follows:

$$f'_k = P^{k-1}\Omega i^Y_{k-1,k} \circ f_{k-1} \cup C(g_k)$$

which makes the right square of the following diagram commutative:

$$\begin{tikzcd}
L_k \arrow[r, "h_k"] \arrow[dd, "g_k"] & X_{k-1} \arrow[r, "i^X_{k-1,k}"] \arrow[d, "f_{k-1}"] & X_k \arrow[dd, dashed, "f'_k"] \\
& P^{k-1}\Omega Y_{k-1} \arrow[d, "P^{k-1}\Omega i^Y_{k-1,k}"] & \\
E^k\Omega Y_k \arrow[r, "p_k^{\Omega Y_k}"] & P^{k-1}\Omega Y_k \arrow[r, "\iota^{\Omega Y_k}_{k-1,k}"] & P^k\Omega Y_k.
\end{tikzcd}$$

By definition, f'_k satisfies the equation,

$$(f'_k \vee \Sigma g_k) \circ \nu_k = \bar{\nu}_k \circ f'_k, \tag{2.1}$$

where $\nu_k : X_k \to X_k \vee \Sigma L_k$ and $\bar{\nu}_k : P^k\Omega Y_k \to P^k\Omega Y_k \vee \Sigma E^k\Omega Y_k$ are the canonical copairings. In the exact sequence $[X_{k-1}, Y_k] \xleftarrow{{i^X_{k-1,k}}^*} [X_k, Y_k] \xleftarrow{q^*} [\Sigma L_k, Y_k]$, we have the equation,

$$\begin{aligned}
{i^X_{k-1,k}}^*(e_k^{Y_k} \circ f'_k) &= e_k^{Y_k} \circ f'_k \circ i^X_{k-1,k} \\
&= e_k^{Y_k} \circ (\iota^{\Omega Y_k}_{k-1,k} \circ P^{k-1}\Omega i^Y_{k-1,k} \circ f_{k-1}) \\
&= e_{k-1}^{Y_k} \circ P^{k-1}\Omega i^Y_{k-1,k} \circ f_{k-1} \\
&= i^Y_{k-1,k} \circ f|_{X_k-1} \\
&= f|_{X_k} \circ i^X_{k-1,k} \\
&= {i^X_{k-1,k}}^*(f|_{X_k}).
\end{aligned}$$

By Theorem B. 10 of [2], there exists a map $\delta'_k : \Sigma L_k \to Y_k$ such that

$$f|_{X_k} = \nabla_{Y_k} \circ (e_k^{Y_k} \circ f'_k \vee \delta'_k) \circ \nu_k.$$

Let us consider the following exact sequence,

$$\begin{tikzcd}
{} \arrow[r] & {[L_k, \Omega P^{k-1}\Omega Y_k]} \arrow[r, "\Omega {e_{k-1}^{Y_k}}_*"] & {[L_k, \Omega Y_k]} \arrow[r, "\Delta_*"] & {[L_k, E^k\Omega Y_k]} \arrow[r] & {} \\
& {[\Sigma L_k, P^{k-1}\Omega Y_k]} \arrow[u, "\mathrm{ad}", "\cong"'] \arrow[r, "{e_{k-1}^{Y_k}}_*"] & {[\Sigma L_k, Y_k].} \arrow[u, "\mathrm{ad}", "\cong"'] & &
\end{tikzcd}$$

Since $\Omega e^{Y_k}_{k-1}$ has a section, there exists a map $\delta_k : \Sigma L_k \to P^{k-1}\Omega Y_k$ such that $\delta'_k = e^{Y_k}_{k-1} \circ \delta_k$. Therefore we have the following equation:

$$\begin{aligned} f|_{X_k} &= \nabla_{Y_k} \circ (e^{Y_k}_k \circ f'_k \vee e^{Y_k}_{k-1} \circ \delta_k) \circ \nu_k \\ &= \nabla_{Y_k} \circ (e^{Y_k}_k \circ f'_k \vee e^{Y_k}_k \circ \iota^{\Omega Y_k}_{k-1,k} \circ \delta_k) \circ \nu_k \\ &= \nabla_{Y_k} \circ (e^{Y_k}_k \vee e^{Y_k}_k) \circ (f'_k \vee \iota^{\Omega Y_k}_{k-1,k} \circ \delta_k) \circ \nu_k \\ &= e^{Y_k}_k \circ \nabla_{P^k\Omega Y_k} \circ (f'_k \vee \iota^{\Omega Y_k}_{k-1,k} \circ \delta_k) \circ \nu_k. \end{aligned}$$

We define a map f_k by a map $\nabla_{P^k\Omega Y_k} \circ (f'_k \vee \iota^{\Omega Y_k}_{k-1,k} \circ \delta_k) \circ \nu_k$, then f_k satisfies the condition of (2). Since ν_k is the copairing, we have the equations

$$pr_1 \circ \nu_k \circ i^X_{k-1,k} = \mathrm{id}_{X_k} \circ i^X_{k-1,k} = i^X_{k-1,k} \quad \text{and} \quad pr_2 \circ \nu_k \circ i^X_{k-1,k} = q \circ i^X_{k-1,k} = 0,$$

where $pr_1 : X_k \vee \Sigma L_k \to X_k$ and $pr_2 : X_k \vee \Sigma L_k \to \Sigma L_k$ are the first and second projections, respectively. Hence we obtain the equation

$$\begin{aligned} f_k \circ i^X_{k-1,k} &= \nabla_{P^k\Omega Y_k} \circ (f'_k \vee \iota^{\Omega Y_k}_{k-1,k} \circ \delta_k) \circ \nu_k \circ i^X_{k-1,k} \\ &= f'_k \circ i^X_{k-1,k} \\ &= \iota^{\Omega Y_k}_{k-1,k} \circ P^{k-1}\Omega i^Y_{k-1,k} \circ f_{k-1}. \end{aligned}$$

It follows that f_k satisfies the condition of (1), too. $\square$

Let $\{f_i : X_i \to P^i\Omega Y_i \,|\, 0 \leq i \leq n\}$ and $\{g_i : L_i \to E^i\Omega Y_i \,|\, 1 \leq i \leq m\}$ be the map obtained from the filtered map $f : X \to Y$ by Proposition 2.3. We denote $\nu_i : X_i \to X_i \vee \Sigma L_i$ and $\bar{\nu}_i : P^i\Omega Y_i \to P^k\Omega Y_i \vee \Sigma E^i\Omega Y_i$ by the canonical copairings.

Proposition 2.4. *If the complex L_i be a co-H-space, then the following diagram is commutative.*

$$\begin{tikzcd} X_i \arrow[r, "\nu_i"] \arrow[d, "f_i"] & X_k \vee \Sigma L_i \arrow[d, "f_i \vee \Sigma g_i"] \\ P^i\Omega Y_i \arrow[r, "\bar{\nu}_i"] & P^i\Omega Y_i \vee \Sigma E^i\Omega Y_i. \end{tikzcd}$$

Proof. By the definition of f_i, and by the relation between the composition and the wedge of maps, we have

$$\begin{aligned} (f_i \vee \Sigma g_i) \circ \nu_i &= \{(\nabla_P \circ (f'_i \vee \iota^{\Omega Y_i}_{i-1,i} \circ \delta_i) \circ \nu_i) \vee \Sigma g_i\} \circ \nu_i \\ &= \{\nabla_P \circ (f'_i \vee \iota^{\Omega Y_i}_{i-1,i} \circ \delta_i) \vee \Sigma g_i\} \circ (\nu_i \vee \mathrm{id}_{\Sigma L_i}) \circ \nu_i \\ &= (\nabla_P \vee \mathrm{id}_E) \circ (f'_i \vee \iota^{\Omega Y_i}_{i-1,i} \circ \delta_i \vee \Sigma g_i) \circ (\nu_i \vee \mathrm{id}_{\Sigma L_i}) \circ \nu_i, \end{aligned}$$

where $\nabla_P = \nabla_{P^i\Omega Y_i}$ and $\mathrm{id}_E = \mathrm{id}_{\Sigma E^i\Omega Y_i}$. Since L_i is the co-H-space, we have the equations

$$\upsilon_i = T \circ \upsilon_i \quad \text{and} \quad (\nu_i \vee \mathrm{id}_{\Sigma L_i}) \circ \nu_i = (\mathrm{id}_{X_i} \vee \upsilon_i) \circ \nu_i,$$

where $\upsilon_i : \Sigma L_i \to \Sigma L_i \vee \Sigma L_i$ is the co-multiplication and $T : \Sigma L_i \vee \Sigma L_i \to \Sigma L_i \vee \Sigma L_i$ is the commutative map. So we can proceed as follows:

$$\begin{aligned}(f_i \vee \Sigma g_i) \circ \nu_i &= (\nabla_P \vee \mathrm{id}_E) \circ (f_i' \vee \iota^{\Omega Y_i}_{i-1,i} \circ \delta_i \vee \Sigma g_i) \circ (\mathrm{id}_{X_i} \vee \upsilon_i) \circ \nu_i \\ &= (\nabla_P \vee \mathrm{id}_E) \circ (f_i' \vee \iota^{\Omega Y_i}_{i-1,i} \circ \delta_i \vee \Sigma g_i) \circ (\mathrm{id}_{X_i} \vee T \circ \upsilon_i) \circ \nu_i \\ &= (\nabla_P \vee \mathrm{id}_E) \circ \{f_i' \vee T' \circ (\Sigma g_i \vee \iota^{\Omega Y_i}_{i-1,i} \circ \delta_i)\} \circ (\mathrm{id}_{X_i} \vee \upsilon_i) \circ \nu_i \\ &= (\nabla_P \vee \mathrm{id}_E) \circ (f_i' \vee T') \\ &\qquad \circ \{\mathrm{id}_{X_i} \vee (\Sigma g_i \vee \iota^{\Omega Y_i}_{i-1,i} \circ \delta_i)\} \circ (\mathrm{id}_{X_i} \vee \upsilon_i) \circ \nu_i \\ &= (\nabla_P \vee \mathrm{id}_E) \circ (\mathrm{id}_P \vee T') \\ &\qquad \circ (f_i' \vee \Sigma g_i \vee \iota^{\Omega Y_i}_{i-1,i} \circ \delta_i) \circ (\nu_i \vee \mathrm{id}_{\Sigma L_i}) \circ \nu_i \\ &= (\nabla_P \vee \mathrm{id}_E) \circ (\mathrm{id}_P \vee T') \circ \{(f_i' \vee \Sigma g_i) \circ \nu_i \vee \iota^{\Omega Y_i}_{i-1,i} \circ \delta_i\} \circ \nu_i,\end{aligned}$$

where $T' : \Sigma E^i\Omega Y_i \vee P^i\Omega Y_i \to P^i\Omega Y_i \vee \Sigma E^i\Omega Y_i$ is the commutative map and $\mathrm{id}_P = \mathrm{id}_{P^i\Omega Y_i}$. By the equation (2.1), we proceed further as follows:

$$\begin{aligned}(f_i \vee \Sigma g_i) \circ \nu_i &= (\nabla_P \vee \mathrm{id}_E) \circ (\mathrm{id}_P \vee T') \circ \{(\bar{\nu}_i \circ f_i' \vee \iota^{\Omega Y_i}_{i-1,i} \circ \delta_i\} \circ \nu_i \\ &= (\nabla_P \vee \mathrm{id}_E) \circ (\mathrm{id}_P \vee T') \circ (\bar{\nu}_i \vee \iota^{\Omega Y_i}_{i-1,i}) \circ (f_i' \vee \delta_i) \circ \nu_i \\ &= (\nabla_P \vee \nabla_{\Sigma E^i\Omega Y_i}) \circ (\mathrm{id}_P \vee T' \vee \mathrm{id}_E) \\ &\qquad \circ (\bar{\nu}_i \vee \bar{\nu}_i) \circ (\mathrm{id}_P \vee \iota^{\Omega Y_i}_{i-1,i}) \circ (f_i' \vee \delta_i) \circ \nu_i \\ &= \bar{\nu}_i \circ \nabla_P \circ (f_i' \vee \iota^{\Omega Y_i}_{i-1,i} \circ \delta_i) \circ \nu_i \\ &= \bar{\nu}_i \circ f_i.\end{aligned}$$

□

3. Cone-Decomposition associated with projective spaces

We denote the k-skeleton of a space X by $(X)^{(k)}$ and the restriction of $f : X \to Y$ on $(X)^{(k)}$ by $(f)^{(k)}$. By the fact that $(f)^{(k)}$ is compressible into $(Y)^{(k)}$, we use the same symbol $(f)^{(k)} : (X)^{(k)} \to (Y)^{(k)}$. And if the dimension of X is less than or equal to n, then we use the same symbol $f : X \to (Y)^{(n)}$, too.

Let G be a compact Lie group with a cone-decomposition of length m, that is, there are cofibration sequences

$$\{K_i \xrightarrow{h_i} F_{i-1} \xrightarrow{i^F_{i-1,i}} F_i \,|\, 1 \leq i \leq m\}, \tag{3.1}$$

with $F_0 = *$ and $F_m \simeq G$. Let l be the dimension of Lie group G.

Lemma 3.1. *Suppose that the complex K_m is the sphere $S^{\ell-1}$ and $\ell \geq 3$, $m \geq 3$. Then there is a cofibre sequence as follows:*

$$(E^m\Omega F_{m-1})^{(\ell-1)} \vee K_m \xrightarrow{p'} (P^{m-1}\Omega F_{m-1})^{(\ell)} \to (P^m\Omega F_m)^{(\ell)}.$$

Proof. First, we determine the homotopy type of the $(\ell-1)$-skeleton of the homotopy fibre of the map $P^{m-1}\Omega i^F_{m-1,m} : P^{m-1}\Omega F_{m-1} \to P^{m-1}\Omega F_m$. Let $\mathfrak{F}$ be

the homotopy fibre of $P^{m-1}\Omega i^F_{m-1,m}$, we consider the following commutative diagram with rows and columns as fibrations:

$$\begin{tikzcd}
\Omega(E^m\Omega F_m, E^m\Omega F_{m-1}) \arrow[r] \arrow[d] & \mathfrak{F} \arrow[r] \arrow[d] & \Omega(F_m, F_{m-1}) \arrow[d] \\
E^m\Omega F_{m-1} \arrow[r, "p_m^{\Omega F_{m-1}}"] \arrow[d, "E^m\Omega i^F_{m-1,m}"] & P^{m-1}\Omega F_{m-1} \arrow[r, "e_{m-1}^{F_{m-1}}"] \arrow[d, "P^{m-1}\Omega i^F_{m-1,m}"] & F_{m-1} \arrow[d, "i^F_{m-1,m}"] \\
E^m\Omega F_m \arrow[r, "p_m^{\Omega F_m}"] & P^{m-1}\Omega F_m \arrow[r, "e_{m-1}^{F_m}"] & F_m.
\end{tikzcd}$$

Since (F_m, F_{m-1}) is $(\ell-1)$-connected, $(\Omega F_m, \Omega F_{m-1})$ is $(\ell-2)$-connected and $(E^m\Omega F_m, E^m\Omega F_{m-1})$ is $(\ell+m-3)$-connected. Hence $\Omega(E^m\Omega F_m, E^m\Omega F_{m-1})$ is $(\ell+m-4)$-connected. By the Serre exact sequence

$$H_{2\ell+m-5}(\Omega(E^m\Omega F_m, E^m\Omega F_{m-1})) \to \cdots \to H_k(\Omega(E^m\Omega F_m, E^m\Omega F_{m-1})) \to$$

$$H_k(\mathfrak{F}) \to H_k(\Omega(F_m, F_{m-1})) \to H_{k-1}(\Omega(E^m\Omega F_m, E^m\Omega F_{m-1})) \to \cdots,$$

we obtain that $H_k(\mathfrak{F})$ is isomorphic to $H_k(\Omega(F_m, F_{m-1}))$ for $k \le \ell \le \ell+m-3$, and hence that $\mathfrak{F}$ is $(\ell-2)$-connected, $\ell \ge 3$. On the other hand, by the Blakers-Massey's theorem, we have $\pi_l(F_m, F_{m-1}) \cong \pi_l(S^l)$, and hence we obtain

$$\pi_{\ell-1}(\Omega(F_m, F_{m-1})) \cong \pi_l(F_m, F_{m-1}) \cong \pi_l(S^l) \cong \mathbb{Z}.$$

Then by Hurewicz Isomorphism Theorem, we obtain

$$H_{\ell-1}(\mathfrak{F}) \cong H_{\ell-1}(\Omega(F_m, F_{m-1})) \cong \pi_{\ell-1}(\Omega(F_m, F_{m-1})) \cong \mathbb{Z}.$$

Thus $\mathfrak{F}$ has the homology decomposition as

$$\mathfrak{F} \simeq S^{\ell-1} \cup \text{ (Moore cells in dimensions } \ge \ell).$$

By Ganea's fibre-cofibre construction (see Ganea [3]), we obtain a map

$$\phi_0 : P^{m-1}\Omega F_{m-1} \cup C\mathfrak{F} \to P^{m-1}\Omega F_m,$$

as the homotopy pushout

$$\begin{tikzcd}
\mathfrak{F} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{HPO}"] & P^{m-1}\Omega F_{m-1} \arrow[d] \\
\{*\} \arrow[r] & P^{m-1}\Omega F_{m-1} \cup C\mathfrak{F},
\end{tikzcd}$$

which has the homotopy type of the homotopy pullback of the diagonal

$$\Delta : P^{m-1}\Omega F_m \to P^{m-1}\Omega F_m \times P^{m-1}\Omega F_m$$

and the inclusion

$$P^{m-1}\Omega F_{m-1} \times P^{m-1}\Omega F_m \cup P^{m-1}\Omega F_m \times \{*\} \hookrightarrow P^{m-1}\Omega F_m \times P^{m-1}\Omega F_m :$$

$$\begin{tikzcd}
P^{m-1}\Omega F_{m-1} \cup C\mathfrak{F} \arrow[r] \arrow[d, "\phi_0"] \arrow[dr, phantom, "\text{HPB}"] & \begin{array}{c} P^{m-1}\Omega F_{m-1} \times P^{m-1}\Omega F_m \\ \cup P^{m-1}\Omega F_m \times \{*\} \end{array} \arrow[d, hook] \\
P^{m-1}\Omega F_m \arrow[r, "\Delta"'] & P^{m-1}\Omega F_m \times P^{m-1}\Omega F_m.
\end{tikzcd}$$

(see, for example, [4, Lemma 2.1] with $(X, A) = (P^{m-1}\Omega F_m, P^{m-1}\Omega F_{m-1})$, $(Y, B) = (P^{m-1}\Omega F_m, \{*\})$ and $Z = P^{m-1}\Omega F_m$). Hence $\mathfrak{F}_0$ is given by the pullback of the trivial map

$$\{*\} \to P^{m-1}\Omega F_m \times P^{m-1}\Omega F_m$$

and the inclusion

$$P^{m-1}\Omega F_{m-1} \times P^{m-1}\Omega F_m \cup P^{m-1}\Omega F_m \times \{*\} \hookrightarrow P^{m-1}\Omega F_m \times P^{m-1}\Omega F_m$$

which has the homotopy type of the pushout

$$\begin{tikzcd}
\mathfrak{F} \times \Omega P^{m-1}\Omega F_m \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{HPO}"] & P^{m-1}\Omega F_{m-1} \arrow[d] \\
\mathfrak{F} \arrow[r] & \mathfrak{F}_0.
\end{tikzcd}$$

(see, for example, [4, Lemma 2.1] with $(X, A) = (P^{m-1}\Omega F_m, P^{m-1}\Omega F_{m-1})$, $(Y, B) = (P^{m-1}\Omega F_m, \{*\})$ and $Z = \{*\}$). Thus the homotopy fibre $\mathfrak{F}_0$ of ϕ_0 has the homotopy type of the join $\mathfrak{F} * \Omega P^{m-1}\Omega F_m$ and is $(\ell-1)$-connected, and hence ϕ_0 is ℓ-connected. Thus we have that

$$(P^{m-1}\Omega F_{m-1})^{(\ell)} \cup CS^{\ell-1} \simeq (P^{m-1}\Omega F_m)^{(\ell)}.$$

We are now ready to show that $(P^m\Omega F_{m-1})^{(\ell)} \cup CS^{\ell-1} \simeq (P^m\Omega F_m)^{(\ell)}$. Since $(E^m\Omega F_m, E^m\Omega F_{m-1})$ is $(\ell + m - 3)$-connected and $m \geq 3$, $(E^m\Omega F_{m-1})^{(\ell-1)} \simeq (E^m\Omega F_m)^{(\ell-1)}$ and hence

$$\begin{aligned}
(P^m\Omega F_{m-1})^{(\ell)} \cup CS^{\ell-1} &\simeq (P^{m-1}\Omega F_{m-1})^{(\ell)} \cup C(S^{\ell-1} \vee (E^m\Omega F_{m-1})^{(\ell-1)}) \\
&\simeq (P^{m-1}\Omega F_m)^{(\ell)} \cup C(E^m\Omega F_{m-1})^{(\ell-1)} \simeq (P^m\Omega F_m)^{(\ell)}.
\end{aligned}$$

This completes the proof of Lemma 3.1. □

Using Lemma 3.1, we construct cone-decompositions of $F_m \times F_1$, $(P^m\Omega F_m)^{(\ell)}$ and $(P^m\Omega F_m)^{(\ell)} \times (\Sigma\Omega F_1)^{(\ell)}$.

First, we construct a cone-decomposition of $F_m{\times}F_1$: Let $K^{m,1}_i$ and $F^{m,1}_i$ be as follows.

$$\begin{aligned}
K^{m,1}_i &= \{K_i{\times}\{*\}\} \vee \{K_{i-1}{*}K_1\} && \text{for } 1 \le i \le m,\\
F^{m,1}_i &= F_i{\times}\{*\} \cup F_{i-1}{\times}F_1 && \text{for } 0 \le i \le m,
\end{aligned}$$

$$K^{m,1}_{m+1} = K_m{*}K_1 \quad\text{and}\quad F^{m,1}_{m+1} = F_m{\times}F_1,$$

where K_0 and F_{-1} are empty sets. We denote a map $\chi_i : (CK_i, K_i) \to (F_i, F_{i-1})$ by the characteristic map. We introduce the relative Whitehead product $[\chi_{i-1}, \chi_1]^r : K_{i-1}{*}K_1 \to F^{m,1}_{i-1}$ defined as follows:

$$\begin{aligned}
&K_{i-1}{*}K_1 = (CK_{i-1}{\times}K_1) \cup (K_{i-1}{\times}CK_1)\\
&\qquad\xrightarrow{(\chi_{i-1}\times\chi_1|_{K_1})\cup(\chi_{i-1}|_{K_{i-1}}\times\chi_1)} F_{i-1}{\times}\{*\} \cup F_{i-2}{\times}F_1 = F^{m,1}_{i-1}.
\end{aligned}$$

Let $w^{m,1}_i : K^{m,1}_i \to F^{m,1}_{i-1}$ be the wedge of maps $(incl) \circ (h_i{\times}\{*\}) : K_i{\times}\{*\} \to F_{i-1}{\times}\{*\} \hookrightarrow F^{m,1}_{i-1}$ and $[\chi_{i-1}, \chi_1]^r$ for $1 \le i \le m$, and $w^{m,1}_{m+1} : K^{m,1}_{m+1} \to F^{m,1}_m$ be $[\chi_m, \mathrm{id}_{\Sigma K_1}]^r$. Let $i^{m,1}_i : F^{m,1}_i \to F^{m,1}_{i+1}$ be the canonical inclusion for $0 \le i \le m$. Then the set of cofibration sequences

$$\{K^{m,1}_i \xrightarrow{w^{m,1}_i} F^{m,1}_{i-1} \xrightarrow{i^{m,1}_{i-1}} F^{m,1}_i \mid 1 \le i \le m+1\} \tag{3.2}$$

is a cone-decomposition of $F_m{\times}F_1$ of length $m+1$.

Second, we construct a cone-decomposition of $(P^m\Omega F_m)^{(\ell)}$. By lemma 3.1, we obtain a cone-decomposition of $(P^m\Omega F_m)^{(\ell)}$ of length m:

$$\left\{\begin{array}{l}
(\Omega F_{m-1})^{(\ell-1)} \to \{*\} \hookrightarrow (\Sigma\Omega F_{m-1})^{(\ell)}\\
(E^2\Omega F_{m-1})^{(\ell-1)} \to (\Sigma\Omega F_{m-1})^{(\ell)} \hookrightarrow (P^2\Omega F_{m-1})^{(\ell)}\\
\qquad\qquad\qquad\vdots\\
(E^{m-1}\Omega F_{m-1})^{(\ell-1)} \to (P^{m-2}\Omega F_{m-1})^{(\ell)} \hookrightarrow (P^{m-1}\Omega F_{m-1})^{(\ell)}\\
(E^m\Omega F_{m-1})^{(\ell-1)} \vee K_m \to (P^{m-1}\Omega F_{m-1})^{(\ell)} \hookrightarrow (P^m\Omega F_m)^{(\ell)}.
\end{array}\right.$$

Third, we construct a cone-decomposition of $(P^m\Omega F_m)^{(\ell)}{\times}(\Sigma\Omega F_1)^{(\ell)}$. Let $\hat{E}_i$ and $\hat{F}_i$ be as follows.

$$\hat{E}_i \quad = \{(E^i\Omega F_{m-1})^{(\ell-1)}{\times}\{*\}\} \vee \{(E^{i-1}\Omega F_{m-1})^{(\ell-1)}{*}(\Omega F_1)^{(\ell-1)}\}$$

for $1 \le i \le m-1$,

$$\begin{aligned}
\hat{E}_m &= \{\{(E^m\Omega F_{m-1})^{(\ell-1)} \vee K_m\}{\times}\{*\}\} \vee \{(E^{m-1}\Omega F_{m-1})^{(\ell-1)}{*}(\Omega F_1)^{(\ell-1)}\},\\
\hat{E}_{m+1} &= \{(E^m\Omega F_{m-1})^{(\ell-1)} \vee K_m\}{*}(\Omega F_1)^{(\ell-1)},\\
\hat{F}_i &= (P^i\Omega F_{m-1})^{(\ell)}{\times}\{*\} \cup (P^{i-1}\Omega F_{m-1})^{(\ell)}{\times}(\Sigma\Omega F_1)^{(\ell)}
\end{aligned}$$

for $0 \le i \le m-1$,

$$\hat{F}_m \quad = (P^m\Omega F_m)^{(\ell)}{\times}\{*\} \cup (P^{m-1}\Omega F_{m-1})^{(\ell)}{\times}(\Sigma\Omega F_1)^{(\ell)}$$

and

$$\hat{F}_{m+1} = (P^m\Omega F_m)^{(\ell)}\times(\Sigma\Omega F_1)^{(\ell)}.$$

Here $E^{-1}\Omega F_{m-1}$ and $P^{-1}\Omega F_{m-1}$ are empty sets. We denote maps

$$\chi' : (C((\Omega F_1)^{(\ell-1)}), (\Omega F_1)^{(\ell-1)}) \to (\Sigma(\Omega F_1)^{(\ell)}, \{*\}),$$

$$\chi'_i : (C(E^i\Omega F_{m-1})^{(\ell-1)}, (E^i\Omega F_{m-1})^{(\ell-1)}) \to ((P^i\Omega F_{m-1})^{(\ell)}, (P^{i-1}\Omega F_{m-1})^{(\ell)})$$

for $0 \leq i \leq m-1$ and

$$\chi'_m : (CE', E') \to ((P^m\Omega F_{m-1})^{(\ell)}, (P^{m-1}\Omega F_{m-1})^{(\ell)})$$

by the characteristic maps, where $E' = (E^m\Omega F_{m-1})^{(\ell-1)} \vee K_m$. Let $\hat{w}_i : \hat{E}_i \to \hat{F}_{i-1}$ be the wedge of maps

$$(incl) \circ ((p_i^{\Omega F_{m-1}})^{(\ell-1)}\times\{*\}) : (E^i\Omega F_{m-1})^{(\ell-1)}\times\{*\} \to (P^{i-1}\Omega F_{m-1})^{(\ell)}\times\{*\} \hookrightarrow \hat{F}_{i-1}$$

and

$$[\chi'_{i-1}, \chi']^r : (E^{i-1}\Omega F_{m-1})^{(\ell-1)}*(\Omega F_1)^{(\ell-1)} \to \hat{F}_{i-1}$$

for $1 \leq i \leq m-1$, $\hat{w}_m : \hat{E}_m \to \hat{F}_{m-1}$ be the wedge of maps

$$(incl) \circ (p'\times\{*\}) : \{(E^m\Omega F_{m-1})^{(\ell-1)} \vee K_m\}\times\{*\} \to (P^{m-1}\Omega F_{m-1})^{(\ell)}\times\{*\} \hookrightarrow \hat{F}_{m-1}$$

and $[\chi'_{m-1}, \chi']^r$, and $\hat{w}_{m+1} : \hat{E}_{m+1} \to \hat{F}_m$ be $[\chi'_m, \chi']^r$, where $p' : (E^m\Omega F_{m-1})^{(\ell-1)} \vee K_m \to (P^{m-1}\Omega F_{m-1})^{(\ell)}$ is the map p' in Lemma 3.1. We denote $\hat{\imath}_i : \hat{F}_i \to \hat{F}_{i+1}$ by the canonical inclusion for $0 \leq i \leq m$. Then the set of cofibration sequences

$$\{\hat{E}_i \xrightarrow{\hat{w}_i} \hat{F}_{i-1} \xrightarrow{\hat{\imath}_{i-1}} \hat{F}_i \mid 1 \leq i \leq m+1\} \tag{3.3}$$

is a cone-decomposition of $(P^m\Omega F_m)^{(\ell)}\times(\Sigma\Omega F_1)^{(\ell)}$ of length $m+1$.

4. Structure map and cone-decomposition

Let a cone-decomposition of F_m be (3.1) and a k-filter of F_m be F_k, we apply this Proposition 2.3 to the identity map $\mathrm{id}_{F_m} : F_m \to F_m$. From this procedure, we obtain the structure maps $\sigma_i : F_i \to P^i\Omega F_i$ for $1 \leq i \leq m$ and the maps $g'_j : K_j \to E^j\Omega F_j$ for $1 \leq j \leq m$. We set $g_j = g'_j : K_j \to (E^j\Omega F_j)^{(\ell-1)}$ for $1 \leq j \leq m-1$ and $g_m : K_j \to (E^m\Omega F_m)^{(\ell-1)} \simeq (E^m\Omega F_{m-1})^{(\ell-1)} \hookrightarrow (E^m\Omega F_{m-1})^{(\ell-1)} \vee K_m$ the composition g'_m and the inclusion map.

Let $\nu_k^{m,1} : F_k^{m,1} \to F_k^{m,1} \vee \Sigma K_k^{m,1}$ and $\hat{\nu}_k : \hat{F}_k \to \hat{F}_k \vee \Sigma\hat{K}_k$ be the canonical copairings for $1 \leq k \leq m+1$. Then,

Lemma 4.1. *the following diagram is commutative:*

$$\begin{tikzcd}
K^{m,1}_{m+1} \arrow[r, "w^{m,1}_{m+1}"] \arrow[d, "g_m * g_1"] & F^{m,1}_m \arrow[r, "i^{m,1}_m"] \arrow[d, "\hat{\sigma}_m"] & F^{m,1}_{m+1} \arrow[r, "\nu^{m,1}_{m+1}"] \arrow[d, "\sigma_m \times \sigma_1"] & F^{m,1}_{m+1} \vee \Sigma K^{m,1}_{m+1} \arrow[d, "\sigma_m \times \sigma_1 \vee \Sigma g_m * g_1"] \\
\hat{E}_{m+1} \arrow[r, "\hat{w}_{m+1}"] & \hat{F}_m \arrow[r, "\hat{i}_m"] & \hat{F}_{m+1} \arrow[r, "\hat{\nu}_{m+1}"] & \hat{F}_{m+1} \vee \Sigma \hat{E}_{m+1}.
\end{tikzcd}$$

Here the map $\hat{\sigma}_m = \sigma_m \times \{*\} \cup \sigma_{m-1} \times \sigma_1$.

To prove this Lemma, it is necessary to show the following equation:

Lemma 4.2.

$$T_1 \circ ((\nu_m \times \mathrm{id}_{F_1}) \vee \mathrm{id}_{\Sigma K^{m,1}_{m+1}}) \circ \nu^{m,1}_{m+1} = (\nu^{m,1}_{m+1} \cup \mathrm{id}_{\Sigma K_m \times F_1}) \circ (\nu_m \times \mathrm{id}_{F_1}).$$

Here $\nu_m : F_m \to F_m \vee \Sigma K_m$ *is the canonical copairing and* $T_1 : F^{m,1}_{m+1} \cup_{F_1} (\Sigma K_m \times F_1) \vee \Sigma K^{m,1}_{m+1} \to (F^{m,1}_{m+1} \vee \Sigma K^{m,1}_{m+1}) \cup_{F_1} (\Sigma K_m \times F_1)$ *is the canonical homeomorphism.*

Proof. First, we show that the following diagram is commutative:

$$\begin{tikzcd}
F^{m,1}_{m+1} \arrow[r, "\nu_m \times \mathrm{id}_{F_1}"] \arrow[d, "\nu^{m,1}_{m+1}"] & F^{m,1}_{m+1} \cup_{F_1} (\Sigma K_m \times F_1) \arrow[d, "\mathrm{id}_{F_m \times F_1} \cup \nu'"] \\
F^{m,1}_{m+1} \vee \Sigma K_m * K_1 & F^{m,1}_{m+1} \cup_{F_1} (\Sigma K_m \times F_1) \vee \Sigma K_m * K_1, \arrow[l, "p_1"']
\end{tikzcd} \tag{4.1}$$

where $\nu' : \Sigma K_m \times F_1 = \Sigma K_m \times \Sigma K_1 \to \Sigma K_m \times \Sigma K_1 \vee \Sigma K_m * K_1$ is the canonical copairing and p_1 is the map pinching $\Sigma K_m \times F_1$ to one point. This follow from Figure 1.
Therefore we have

$$\begin{aligned} & T_1 \circ ((\nu_m \times \mathrm{id}_{F_1}) \vee \mathrm{id}_{\Sigma K^{m,1}_{m+1}}) \circ \nu^{m,1}_{m+1} \\ & \qquad = T_1 \circ ((\nu_m \times \mathrm{id}_{F_1}) \vee \mathrm{id}_{\Sigma K^{m,1}_{m+1}}) \circ p_1 \circ (\mathrm{id}_{F^{m,1}_{m+1}} \cup \nu') \circ (\nu_m \times \mathrm{id}_{F_1}). \end{aligned}$$

Let us denote $p_2 : F^{m,1}_{m+1} \cup_{F_1} (\Sigma K_m \times F_1) \cup_{F_1} (\Sigma K_m \times F_1) \vee \Sigma K^{m,1}_{m+1} \to F^{m,1}_{m+1} \cup_{F_1} (\Sigma K_m \times F_1) \vee \Sigma K^{m,1}_{m+1}$ by the map pinching the second $\Sigma K_m \times F_1$ to one point, $p_3 : F^{m,1}_{m+1} \cup_{F_1} ((\Sigma K_m \times F_1) \vee \Sigma K^{m,1}_{m+1}) \cup_{F_1} (\Sigma K_m \times F_1) \to (F^{m,1}_{m+1} \vee \Sigma K^{m,1}_{m+1}) \cup_{F_1} \Sigma K^{m,1}_{m+1}$ by the map pinching the first $\Sigma K_m \times F_1$ to one point, $\nu_0 : \Sigma K_m \to \Sigma K_m \vee \Sigma K_m$ by the canonical co-multiplication and $T_0 : \Sigma K_m \vee \Sigma K_m \to \Sigma K_m \vee \Sigma K_m$ by the

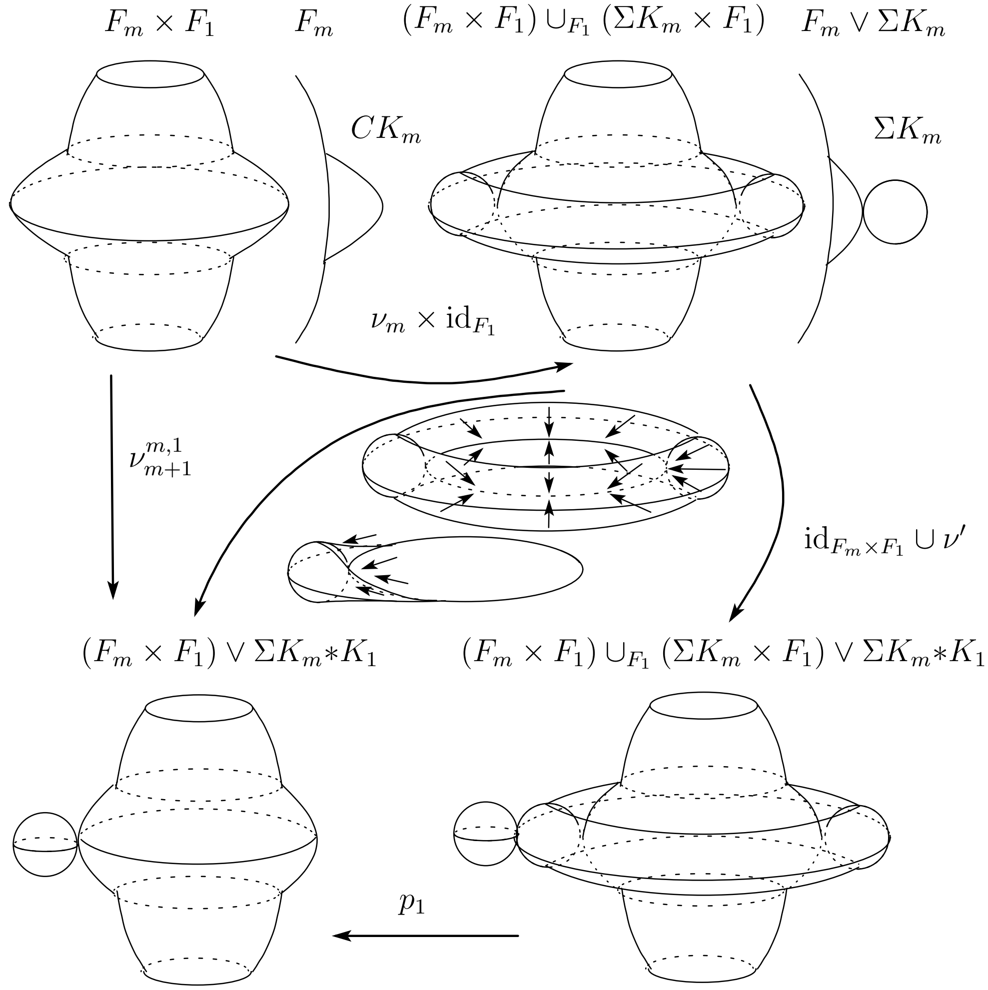

Figure 1.
@

commutative map. It is easy to check the following:

$$
\begin{aligned}
T_1 \circ ((\nu_m {\times} \mathrm{id}_{F_1}) \vee \mathrm{id}_{\Sigma K^{m,1}_{m+1}}) \circ \nu^{m,1}_{m+1} \\
&= T_1 \circ p_2 \circ ((\nu_m {\times} \mathrm{id}_{F_1}) \cup \mathrm{id}_{\Sigma K_m \times F_1} \vee \mathrm{id}_{\Sigma K_m * K_1}) \\
&\qquad \circ (\mathrm{id}_{F^{m,1}_{m+1}} \cup \nu') \circ (\nu_m {\times} \mathrm{id}_{F_1}) \\
&= T_1 \circ p_2 \circ (\mathrm{id}_{F^{m,1}_{m+1}} \cup \mathrm{id}_{\Sigma K_m \times F_1} \cup \nu') \\
&\qquad \circ ((\nu_m {\times} \mathrm{id}_{F_1}) \cup \mathrm{id}_{\Sigma K_m \times F_1}) \circ (\nu_m {\times} \mathrm{id}_{F_1}) \\
&= p_3 \circ (\mathrm{id}_{F^{m,1}_{m+1}} \cup \nu' \cup \mathrm{id}_{\Sigma K_m \times F_1}) \circ (\mathrm{id}_{F^{m,1}_{m+1}} \cup (T_0 {\times} \mathrm{id}_{F_1})) \\
&\qquad \circ ((\nu_m {\times} \mathrm{id}_{F_1}) \cup \mathrm{id}_{\Sigma K_m \times F_1}) \circ (\nu_m {\times} \mathrm{id}_{F_1}).
\end{aligned}
$$

Using the equations $(\mathrm{id}_{F_m}\times\nu_0)\circ\nu_m=(\nu_m\times\mathrm{id}_{F_m})\circ\nu_m$ and $T_0\circ\nu_0=\nu_0$ from the assumption that K_m is a co-H-space, we have

$$
\begin{aligned}
&T_1\circ((\nu_m\times\mathrm{id}_{F_1})\vee\mathrm{id}_{\Sigma K^{m,1}_{m+1}})\circ\nu^{m,1}_{m+1}\\
&\qquad=p_3\circ(\mathrm{id}_{F^{m,1}_{m+1}}\cup\nu'\cup\mathrm{id}_{\Sigma K_m\times F_1})\circ(\mathrm{id}_{F^{m,1}_{m+1}}\cup(T_0\times\mathrm{id}_{F_1}))\\
&\qquad\qquad\circ(\mathrm{id}_{F^{m,1}_{m+1}}\cup(\nu_0\times\mathrm{id}_{F_1}))\circ(\nu_m\times\mathrm{id}_{F_1})\\
&\qquad=p_3\circ(\mathrm{id}_{F^{m,1}_{m+1}}\cup\nu'\cup\mathrm{id}_{\Sigma K_m\times F_1})\\
&\qquad\qquad\circ(\mathrm{id}_{F^{m,1}_{m+1}}\cup(\nu_0\times\mathrm{id}_{F_1}))\circ(\nu_m\times\mathrm{id}_{F_1})\\
&\qquad=p_3\circ(\mathrm{id}_{F^{m,1}_{m+1}}\cup\nu'\cup\mathrm{id}_{\Sigma K_m\times F_1})\\
&\qquad\qquad\circ((\nu_m\times\mathrm{id}_{F_1})\cup\mathrm{id}_{\Sigma K_m\times F_1})\circ(\nu_m\times\mathrm{id}_{F_1}).
\end{aligned}
$$

Using the diagram (4.1), we proceed further as follows:

$$T_1\circ((\nu_m\times\mathrm{id}_{F_1})\vee\mathrm{id}_{\Sigma K^{m,1}_{m+1}})\circ\nu^{m,1}_{m+1}=(\nu^{m,1}_{m+1}\cup\mathrm{id}_{\Sigma K_m\times F_1})\circ(\nu_m\times\mathrm{id}_{F_1}).$$

This completes the proof of Lemma 4.2. □

Proof of Lemma 4.1. The commutativity of the left square follows from Proposition 2.9 of [11]. It is obvious that the middle square is commutative. We show the equation $(\sigma_m\times\sigma_1\vee\Sigma g_m{*}g_1)\circ\nu^{m,1}_{m+1}=\hat{\nu}_{m+1}\circ(\sigma_m\times\sigma_1)$. Recall that the construction of the structure map $\sigma_m:F_m\to P^m\Omega F_m$, we can see that $\sigma_m=\nabla_{P^m\Omega F_m}\circ(\sigma'_m\vee\iota^{\Omega F_m}_{m-1,m}\circ\delta_m)\circ\nu_m$. Here σ'_m is the induced map from the following diagram;

$$
\begin{tikzcd}
K_m \arrow[r, "h_m"] \arrow[d, "g'_m"] & F_{m-1} \arrow[r, "i^F_{m-1,m}"] \arrow[d, "P^{m-1}\Omega i^F_{m-1,m}\circ\sigma_{m-1}"] & F_m \arrow[d, dashed, "\sigma'_m"] \\
E^m\Omega F_m \arrow[r, "P^{\Omega F_m}_m"] & P^{m-1}\Omega F_m \arrow[r, "\iota^{\Omega F_m}_{m-1,m}"] & P^m\Omega F_m,
\end{tikzcd}
$$

and $\delta_m:\Sigma K_m\to P^{m-1}\Omega F_m$ is the map pulled back the difference map $\delta'_m:\Sigma K_m\to F_m$ which is the difference between the identity map of F_m and $e^{F_m}_m\circ\sigma'_m$.

So we have the equation:

$$\begin{aligned}
&(\sigma_m{\times}\sigma_1 \vee \Sigma g_m{*}g_1) \circ \nu^{m,1}_{m+1}\\
&\quad = \{(\nabla_{P^m\Omega F_m} \circ (\sigma'_m \vee (\iota^{\Omega F_m}_{m-1,m} \circ \delta_m)) \circ \nu_m){\times}\sigma_1 \vee \Sigma g_m{*}g_1\} \circ \nu^{m,1}_{m+1}\\
&\quad = \{(\nabla_{P^m\Omega F_m}{\times}\mathrm{id}_{\Sigma\Omega F_1}) \circ ((\sigma'_m \vee (\iota^{\Omega F_m}_{m-1,m} \circ \delta_m)){\times}\sigma_1)\\
&\qquad\quad \circ (\nu_m{\times}\mathrm{id}_{F_1}) \vee \Sigma g_m{*}g_1\} \circ \nu^{m,1}_{m+1}\\
&\quad = (\nabla_{P^m\Omega F_m}{\times}\mathrm{id}_{\Sigma\Omega F_1} \vee \mathrm{id}_{\Sigma\hat{E}_{m+1}})\\
&\qquad\quad \circ \{((\sigma'_m \vee (\iota^{\Omega F_m}_{m-1,m} \circ \delta_m)){\times}\sigma_1) \vee \Sigma g_m{*}g_1\}\\
&\qquad\quad \circ ((\nu_m{\times}\mathrm{id}_{F_1}) \vee \mathrm{id}_{\Sigma K^{m,1}_{m+1}}) \circ \nu^{m,1}_{m+1}\\
&\quad = (\nabla_{P^m\Omega F_m}{\times}\mathrm{id}_{\Sigma\Omega F_1} \vee \mathrm{id}_{\Sigma\hat{E}_{m+1}})\\
&\qquad\quad \circ \{(\sigma'_m{\times}\sigma_1) \cup ((\iota^{\Omega F_m}_{m-1,m} \circ \delta_m){\times}\sigma_1) \vee \Sigma g_m{*}g_1\}\\
&\qquad\quad \circ ((\nu_m{\times}\mathrm{id}_{F_1}) \vee \mathrm{id}_{\Sigma K^{m,1}_{m+1}}) \circ \nu^{m,1}_{m+1}\\
&\quad = (\nabla_{P^m\Omega F_m}{\times}\mathrm{id}_{\Sigma\Omega F_1} \vee \mathrm{id}_{\Sigma\hat{E}_{m+1}})\\
&\qquad\quad \circ T_2 \circ \{(\sigma'_m{\times}\sigma_1 \vee \Sigma g_m{*}g_1) \cup ((\iota^{\Omega F_m}_{m-1,m} \circ \delta_m){\times}\sigma_1)\}\\
&\qquad\quad \circ T_1 \circ ((\nu_m{\times}\mathrm{id}_{F_1}) \vee \mathrm{id}_{\Sigma K^{m,1}_{m+1}}) \circ \nu^{m,1}_{m+1},
\end{aligned}$$

where $T_2 : (\hat{F}_{m+1} \vee \Sigma\hat{E}_{m+1}) \cup_{\Sigma\Omega F_1} \hat{F}_{m+1} \to (\hat{F}_{m+1} \cup_{\Sigma\Omega F_1} \hat{F}_{m+1}) \vee \Sigma\hat{E}_{m+1}$ is the canonical homeomorphism. By Lemma 4.2, we can proceed as follows:

$$\begin{aligned}
&(\sigma_m{\times}\sigma_1 \vee \Sigma g_m{*}g_1) \circ \nu^{m,1}_{m+1}\\
&\quad = (\nabla_{P^m\Omega F_m}{\times}\mathrm{id}_{\Sigma\Omega F_1} \vee \mathrm{id}_{\Sigma\hat{E}_{m+1}})\\
&\qquad\quad \circ T_2 \circ \{(\sigma'_m{\times}\sigma_1 \vee \Sigma g_m{*}g_1) \cup ((\iota^{\Omega F_m}_{m-1,m} \circ \delta_m){\times}\sigma_1)\}\\
&\qquad\quad \circ (\nu^{m,1}_{m+1} \cup \mathrm{id}_{\Sigma K_m \times F_1}) \circ (\nu_m{\times}\mathrm{id}_{F_1})\\
&\quad = (\nabla_{P^m\Omega F_m}{\times}\mathrm{id}_{\Sigma\Omega F_1} \vee \mathrm{id}_{\Sigma\hat{E}_{m+1}}) \circ T_2\\
&\qquad\quad \circ \{((\sigma'_m{\times}\sigma_1 \vee \Sigma g_m{*}g_1) \circ \nu^{m,1}_{m+1}) \cup ((\iota^{\Omega F_m}_{m-1,m} \circ \delta_m){\times}\sigma_1)\}\\
&\qquad\quad \circ (\nu_m{\times}\mathrm{id}_{F_1}).
\end{aligned}$$

By the definitions of σ'_m and σ_1, we have

$$\begin{aligned}
&(\sigma_m{\times}\sigma_1 \vee \Sigma g_m{*}g_1) \circ \nu^{m,1}_{m+1}\\
&\quad = (\nabla_{P^m\Omega F_m}{\times}\mathrm{id}_{\Sigma\Omega F_1} \vee \mathrm{id}_{\Sigma\hat{E}_{m+1}}) \circ T_2\\
&\qquad\quad \circ \{(\hat{\nu}_{m+1} \circ (\sigma'_m{\times}\sigma_1)) \cup ((\iota^{\Omega F_m}_{m-1,m} \circ \delta_m){\times}\sigma_1)\}\\
&\qquad\quad \circ (\nu_m{\times}\mathrm{id}_{F_1})\\
&\quad = (\nabla_{P^m\Omega F_m}{\times}\mathrm{id}_{\Sigma\Omega F_1} \vee \nabla_{\Sigma\hat{E}_{m+1}}) \circ T_3\\
&\qquad\quad \circ \{\hat{\nu}_{m+1} \circ (\sigma'_m{\times}\sigma_1) \cup i_1 \circ ((\iota^{\Omega F_m}_{m-1,m} \circ \delta_m){\times}\sigma_1)\}\\
&\qquad\quad \circ (\nu_m{\times}\mathrm{id}_{F_1}).
\end{aligned}$$

Here $i_1 : \hat{F}_{m+1} \to \hat{F}_{m+1} \vee \Sigma\hat{E}_{m+1}$ is the inclusion map and $T_3 : (\hat{F}_{m+1} \vee \Sigma\hat{E}_{m+1}) \cup_{\Sigma\Omega F_1} (\hat{F}_{m+1} \vee \Sigma\hat{E}_{m+1}) \to (\hat{F}_{m+1} \cup_{\Sigma\Omega F_1} \hat{F}_{m+1}) \vee \Sigma\hat{E}_{m+1} \vee \Sigma\hat{E}_{m+1}$ is the canonical homeomorphism.

$$
\begin{aligned}
&(\sigma_m \times \sigma_1 \vee \Sigma g_m * g_1) \circ \nu^{m,1}_{m+1} \\
&\quad = (\nabla_{P^m\Omega F_m} \times \mathrm{id}_{\Sigma\Omega F_1} \vee \nabla_{\Sigma\hat{E}_{m+1}}) \circ T_3 \circ (\hat{\nu}_{m+1} \cup \hat{\nu}_{m+1}) \\
&\qquad \circ \{(\sigma'_m \times \sigma_1) \cup ((\iota^{\Omega F_m}_{m-1,m} \circ \delta_m) \times \sigma_1)\} \circ (\nu_m \times \mathrm{id}_{F_1}) \\
&\quad = \hat{\nu}_{m+1} \circ (\nabla_{P^m\Omega F_m} \times \mathrm{id}_{\Sigma\Omega F_1}) \\
&\qquad \circ \{(\sigma'_m \times \sigma_1) \cup ((\iota^{\Omega F_m}_{m-1,m} \circ \delta_m) \times \sigma_1)\} \circ (\nu_m \times \mathrm{id}_{F_1}) \\
&\quad = \hat{\nu}_{m+1} \circ (\nabla_{P^m\Omega F_m} \times \mathrm{id}_{\Sigma\Omega F_1}) \\
&\qquad \circ \{(\sigma'_m \vee (\iota^{\Omega F_m}_{m-1,m} \circ \delta_m)) \times \sigma_1\} \circ (\nu_m \times \mathrm{id}_{F_1}) \\
&\quad = \hat{\nu}_{m+1} \circ \{\nabla_{P^m\Omega F_m} \circ (\sigma'_m \vee (\iota^{\Omega F_m}_{m-1,m} \circ \delta_m)) \circ \nu_m \times \sigma_1\} \\
&\quad = \hat{\nu}_{m+1} \circ (\sigma_m \times \sigma_1).
\end{aligned}
$$

This completes the proof. □

5. Proof of Theorem 1.4

In the fibre sequence $G \hookrightarrow E \to \Sigma A$, by the James-Whitehead decomposition (see Theorem VII.(1.15) of Whitehead [14]), the total space E has the homotopy type of the space $G \cup_\psi G \times CA$. Here ψ is the following composition:

$$\psi : G \times A \xrightarrow{\mathrm{id}_G \times \alpha} G \times G \xrightarrow{\mu} G.$$

Since $G \simeq F_m$ and α is compressible into F'_1, we can see that

$$\psi : G \times A \simeq F_m \times A \xrightarrow{\mathrm{id}_{F_m} \times \alpha} F_m \times F'_1 \subset F_m \times F_1 \subset F_m \times F_m \simeq G \times G \xrightarrow{\mu} G \simeq F_m$$

and E is the homotopy push out of the following sequence:

$$F_m \xleftarrow{pr_1} F_m \times A \xrightarrow{\mathrm{id}_{F_m} \times \alpha} F_m \times F_1 \xrightarrow{\mu|_{F_m \times F_1}} F_m.$$

We construct spaces and maps such that the homotopy push out of these maps dominates E.

The condition of $H_1(\alpha) = 0$ implies that

$$\Sigma\,\mathrm{ad}(\alpha) = \sigma_1|_{F'_1} \circ \alpha : A \to F'_1 \to \Sigma\Omega F'_1. \tag{5.1}$$

We denote $\mu_{i,j} : F_i \times F_j \to F_m$ by the restriction of $\mu : G \times G \to G$ to $F_i \times F_j \subset F_m \times F_m \simeq G \times G$ for $i, j \leq m$. Then

Lemma 5.1. *the following diagram is commutative:*

$$\begin{tikzcd}
F_m & F_m{\times}A \arrow[l, "pr_1"'] \arrow[r, "\mathrm{id}_{F_m}{\times}\alpha"] \arrow[d, "\sigma_m{\times}\sigma_A"] & F_m{\times}F_1 \arrow[r, "\mu_{m,1}"] \arrow[d, "\sigma_m{\times}\sigma_1"] & F_m \arrow[d, "\iota^{\Omega F_m}_{m,m+1}\circ\sigma_m"] \\
P^{m+1}\Omega F_m \arrow[d, "e^{F_m}_{m+1}"] & (P^m\Omega F_m)^{(\ell)}{\times}(\Sigma\Omega A)^{(\ell)} \arrow[l, "\phi"'] \arrow[r, "\chi"] \arrow[d, "(e^{F_m}_m)^{(\ell)}{\times}(e^A_1)^{(\ell)}"] & \hat{F}_{m+1} \arrow[d, "(e^{F_m}_m)^{(\ell)}{\times}(e^{F_1}_1)^{(\ell)}"] & P^{m+1}\Omega F_m \arrow[d, "e^{F_m}_{m+1}"] \\
F_m & F_m{\times}A \arrow[l, "pr_1"'] \arrow[r, "\mathrm{id}_{F_m}{\times}\alpha"] & F_m{\times}F_1 \arrow[r, "\mu_{m,1}"] & F_m.
\end{tikzcd}$$

with the left column's top arrow $F_m \xrightarrow{\iota^{\Omega F_m}_{m,m+1}\circ\sigma_m} P^{m+1}\Omega F_m$.

Here the map ϕ and χ are $(\iota^{\Omega F_m}_{m,m+1})^{(\ell)} \circ pr_1$ and $\mathrm{id}_{(P^m\Omega F_m)^{(\ell)}}{\times}(\Sigma\Omega\alpha)^{(\ell)}$, respectively.

Proof. It is obvious that the top left square is commutative. By the equation $e^{F_m}_m = e^{F_m}_{m+1} \circ \iota^{\Omega F_m}_{m,m+1}$, the bottom left square is commutative. The commutativity of the bottom middle square follows from the equation $\alpha \circ e^A_1 = e^{F_1}_1 \circ P^1\Omega\alpha = e^{F_1}_1 \circ \Sigma\Omega\alpha$. By the equation (5.1), we have the commutative diagram:

$$\begin{tikzcd}
F_m{\times}A \arrow[rr, "\mathrm{id}_{F_m}{\times}\alpha"] \arrow[d, "\sigma_m{\times}\sigma_A"] \arrow[drr, "\sigma_m{\times}\Sigma\,\mathrm{ad}(\alpha)"] & & F_m{\times}F'_1 \arrow[r, "\mathrm{id}_{F_m}{\times}i'"] \arrow[d, "\sigma_m{\times}\sigma_1|_{F'_1}"] & F_m{\times}F_1 \arrow[d, "\sigma_m{\times}\sigma_1"] \\
P^m\Omega F_m{\times}\Sigma\Omega A \arrow[rr, "\mathrm{id}_{P^m\Omega F_m}{\times}\Sigma\Omega\alpha"] & & P^m\Omega F_m{\times}\Sigma\Omega F'_1 \arrow[r, "\mathrm{id}_{P^m\Omega F_m}{\times}\Sigma\Omega i'"] & \hat{F}_{m+1},
\end{tikzcd}$$

where σ_A is the evaluation map and i' is the inclusion map. Thus, the top middle square is commutative. Since σ_m and σ_1 satisfy the condition (2) of Proposition 2.3, we have $e^{F_m}_m \circ \sigma_m = \mathrm{id}_{F_m}$, $e^{F_1}_1 \circ \sigma_1 = \mathrm{id}_{F_1}$ and $e^{F_m}_{m+1} \circ \iota^{\Omega F_m}_{m,m+1} \circ \sigma_m = e^{F_m}_m \circ \sigma_m = \mathrm{id}_{F_m}$. Therefore the right rectangular is commutative, too. □

Lemma 5.2. *In the diagram of Lemma 5.1, there is a map $\hat{\mu} : \hat{F}_{m+1} \to P^{m+1}\Omega F_m$ such that the right rectangular diagram is commutative.*

Proof. First, we construct a map $\hat{\mu}_k : \hat{F}_k \to P^k\Omega F_m$. Let a cone-decomposition of $F_m{\times}F_1$ be (3.2), a cone-decomposition of $\hat{F}_{m+1}$ be (3.3) and a k-filter of F_m be F_m for all k. Let us consider that the restriction of $(e^{F_m}_m)^{(\ell)}{\times}(e^{F_1}_1)^{(\ell)}$ on $\hat{F}_k$ is

$$(e^{F_{m-1}}_k)^{(\ell)}{\times}\{*\} \cup (e^{F_{m-1}}_{k-1})^{(\ell)}{\times}(e^{F_1}_1)^{(\ell)} : \hat{F}_k \to F_m{\times}F_1,$$

then the map $\mu_{m,1} \circ \{(e^{F_m}_m)^{(\ell)}{\times}(e^{F_1}_1)^{(\ell)}\} : \hat{F}_{m+1} \to F_m{\times}F_1 \to F_m$ is a filtered map. Applying this filtered map to Proposition 2.3, we obtain the map

$$\hat{\mu}_k : \hat{F}_k \to P^k\Omega F_m$$

for $0 \le k \le m+1$.

Second, for $0 \le k \le m$, we assert that the equation of maps

$$\iota^{\Omega F_m}_{m,m+1} \circ \sigma_m \circ \mu^{m,1}_k = \iota^{\Omega F_m}_{k,m+1} \circ \hat{\mu}_k \circ j_k \circ \hat{\sigma}_k : F^{m,1}_k \to P^{m+1}\Omega F_m, \tag{5.2}$$

where $\mu^{m,1}_k = \mu_{k,0} \cup \mu_{k-1,1} : F^{m,1}_k = F_k{\times}\{*\} \cup F_{k-1}{\times}F_1 \to F_m$,

$\hat{\sigma}_k = \sigma_k{\times}\{*\} \cup \sigma_{k-1}{\times}\sigma_1 : F^{m,1}_k \to (P^k\Omega F_k)^{(\ell)}{\times}\{*\} \cup (P^{k-1}\Omega F_{k-1})^{(\ell)}{\times}(\Sigma\Omega F_1)^{(\ell)}$

and $j_t = (P^t\Omega i^F_{t,m-1})^{(\ell)}\times\{*\}\cup(P^{t-1}\Omega i^F_{t-1,m-1}))^{(\ell)}\times\mathrm{id}_{(\Sigma\Omega F_1)^{(\ell)}}$ for $1\le t\le m-1$ and $j_m=\mathrm{id}_{\hat F_m}$. Note that this condition is natural to cone-decompositions. This is proved by induction on k. The case $k=0$ is clear, since both maps are constant maps. Assume the kth of (5.2). Let us consider the cofibre sequence $K^{m,1}_{k+1}\xrightarrow{w^{m,1}_{k+1}}F^{m,1}_k\xrightarrow{i^{m,1}_k}F^{m,1}_{k+1}$. Since σ_i satisfy the condition (1) of Proposition 2.3, the following diagram is commutative

$$\begin{tikzcd}
F_i \arrow[r, "\sigma_i"] \arrow[d, "i^F_{i,i+1}"] & P^i\Omega F_i \arrow[r, "P^i\Omega i^F_{i,i+1}"] & P^i\Omega F_{i+1} \arrow[r, "P^i\Omega i^F_{i+1,m-1}"] \arrow[d, "\iota^{\Omega F_{i+1}}_{i,i+1}"] & P^i\Omega F_{m-1} \arrow[d, "\iota^{\Omega F_{m-1}}_{i,i+1}"] \\
F_{i+1} \arrow[rr, "\sigma_{i+1}"] & & P^{i+1}\Omega F_{i+1} \arrow[r, "P^{i+1}\Omega i^F_{i+1,m-1}"] & P^{i+1}\Omega F_{m-1}
\end{tikzcd}$$

for $1\le i\le m-1$. So we have $j_{k+1}\circ\hat\sigma_{k+1}\circ i^{m,1}_k=\hat{i}_k\circ j_k\circ\hat\sigma_k$. By the condition (1) of Proposition 2.3 of $\hat\mu_{k+1}$, we obtain $\hat\mu_{k+1}\circ\hat{i}_k=\iota^{\Omega F_m}_{k,m}\circ\hat\mu_k$. Thus we have the equation

$$\begin{aligned}
i^{m,1*}_k(\iota^{\Omega F_m}_{k+1,m}\circ\hat\mu_{k+1}\circ j_{k+1}\circ\hat\sigma_{k+1}) &= \iota^{\Omega F_m}_{k+1,m}\circ\hat\mu_{k+1}\circ j_{k+1}\circ\hat\sigma_{k+1}\circ i^{m,1}_k\\
&= \iota^{\Omega F_m}_{k+1,m}\circ\hat\mu_{k+1}\circ\hat{i}_k\circ j_k\circ\hat\sigma_k\\
&= \iota^{\Omega F_m}_{k,m}\circ\hat\mu_k\circ j_k\circ\hat\sigma_k.
\end{aligned}$$

By the induction hypothesis, we proceed further as follows:

$$\begin{aligned}
i^{m,1*}_k(\iota^{\Omega F_m}_{k+1,m}\circ\hat\mu_{k+1}\circ j_{k+1}\circ\hat\sigma_k) &= \sigma_m\circ\mu^{m,1}_{k+1}\circ i^{m,1}_k\\
&= i^{m,1*}_k(\sigma_m\circ\mu^{m,1}_{k+1}).
\end{aligned}$$

By Theorem B. 10 of [2], there exists a map $\delta_{k+1}:\Sigma K^{m,1}_{k+1}\to P^m\Omega F_m$ such that

$$\sigma_m\circ\mu^{m,1}_{k+1}=\nabla_{P^m\Omega F_m}\circ(\iota^{\Omega F_m}_{k+1,m}\circ\hat\mu_{k+1}\circ j_{k+1}\circ\hat\sigma_{k+1}\vee\delta_{k+1})\circ\nu^{m,1}_{k+1}. \tag{5.3}$$

By the condition (2) of Proposition 2.3 of $\hat\mu_{k+1}$, we have the equation

$$e^{F_m}_m\circ\iota^{\Omega F_m}_{k+1,m}\circ\hat\mu_{k+1}=e^{F_m}_{k+1}\circ\hat\mu_{k+1}=\mu_{m,1}\circ\{(e^{F_{m-1}}_{k+1})^{(\ell)}\times\{*\}\cup(e^{F_{m-1}}_k)^{(\ell)}\times(e^{F_1}_1)^{(\ell)}\}.$$

By the commutative diagram

$$\begin{tikzcd}
F_i \arrow[r, "\sigma_i"] \arrow[dr, "\mathrm{id}_{F_i}"'] & (P^i\Omega F_i)^{(\ell)} \arrow[r, "(P^i\Omega i^F_{i,m-1})^{(\ell)}"] \arrow[d, "(e^{F_i}_i)^{(\ell)}"] & (P^i\Omega F_{m-1})^{(\ell)} \arrow[r, "(e^{F_{m-1}}_i)^{(\ell)}"] & F_{m-1} \\
& F_i \arrow[urr, "i^F_{i,m-1}"'] & &
\end{tikzcd}$$

for $i=k,\ k+1\le m-1$ and by the maps $\sigma_m\circ(e^{F_m}_m)^{(\ell)}$ and j_m are equal to identity maps up to homotopy, we have the equation

$$\{(e^{F_{m-1}}_{k+1})^{(\ell)}\times\{*\}\cup(e^{F_{m-1}}_k)^{(\ell)}\times(e^{F_1}_1)^{(\ell)}\}\circ j_{k+1}\circ\hat\sigma_{k+1}=i^{m,1}_{k+1}.$$

Thus we obtain

$$\begin{aligned} e^{F_m}_m \circ \iota^{\Omega F_m}_{k+1,m} \circ \hat{\mu}_{k+1} \circ j_{k+1} \circ \hat{\sigma}_{k+1} &= \mu_{m,1} \circ i^{m,1}_{k+1} = \mu^{m,1}_{k+1} \\ &= e^{F_m}_m \circ \sigma_m \circ \mu^{m,1}_{k+1} \end{aligned}$$

and

$$\begin{aligned} e^{F_m}_m \circ \sigma_m \circ \mu^{m,1}_{k+1} &= e^{F_m}_m \circ \nabla_{P^m\Omega F_m} \circ (\iota^{\Omega F_m}_{k+1,m} \circ \hat{\mu}_{k+1} \circ j_{k+1} \circ \hat{\sigma}_{k+1} \vee \delta_{k+1}) \circ \nu^{m,1}_{k+1} \\ &= \nabla_{F_m} \circ (e^{F_m}_m \circ \iota^{\Omega F_m}_{k+1,m} \circ \hat{\mu}_{k+1} \circ j_{k+1} \circ \hat{\sigma}_{k+1} \vee e^{F_m}_m \circ \delta_{k+1}) \circ \nu^{m,1}_{k+1} \\ &= \nabla_{F_m} \circ (e^{F_m}_m \circ \sigma_m \circ \mu^{m,1}_{k+1} \vee e^{F_m}_m \circ \delta_{k+1}) \circ \nu^{m,1}_{k+1}. \end{aligned}$$

Using Theorem 2.7 (1) of [9] and the multiplication μ on $G \simeq F_m$, the map $e^{F_m}_m \circ \delta_{k+1} : \Sigma K^{m,1}_{k+1} \to F_m$ is null-homotopic. Using the following exact sequence,

$$\cdots \to [\Sigma K^{m,1}_{k+1}, E^{m+1}\Omega F_m] \xrightarrow{p^{\Omega F_m}_{m+1}{}_*} [\Sigma K^{m,1}_{k+1}, P^m\Omega F_m] \xrightarrow{e^{F_m}_m{}_*} [\Sigma K^{m,1}_{k+1}, F_m].$$

By the equation $e^{F_m}_m \circ \delta_{k+1} = 0$, there exists a map $\delta'_{k+1} : \Sigma K^{m,1}_{k+1} \to E^{m+1}\Omega F_m$ such that $\delta_{k+1} = p^{\Omega F_m}_{m+1} \circ \delta'_{k+1}$. Since $E^{m+1}\Omega F_m \xrightarrow{p^{\Omega F_m}_{m+1}} P^m\Omega F_m \xrightarrow{\iota^{\Omega F_m}_{m,m+1}} P^{m+1}\Omega F_m$ is the cofibre sequence, we have $\iota^{\Omega F_m}_{m,m+1} \circ \delta_{k+1} = \iota^{\Omega F_m}_{m,m+1} \circ p^{\Omega F_m}_{m+1} \circ \delta'_{k+1} = 0$ and

$$\begin{aligned} &\iota^{\Omega F_m}_{m,m+1} \circ \nabla_{P^m\Omega F_m} \circ (\iota^{\Omega F_m}_{k+1,m} \circ \hat{\mu}_{k+1} \circ j_{k+1} \circ \hat{\sigma}_{k+1} \vee \delta_{k+1}) \circ \nu^{m,1}_{k+1} \\ &\quad = \nabla_{P^{m+1}\Omega F_m} \circ (\iota^{\Omega F_m}_{k+1,m+1} \circ \hat{\mu}_{k+1} \circ j_{k+1} \circ \hat{\sigma}_{k+1} \vee \iota^{\Omega F_m}_{m,m+1} \circ \delta_{k+1}) \circ \nu^{m,1}_{k+1} \\ &\quad = \nabla_{P^{m+1}\Omega F_m} \circ (\iota^{\Omega F_m}_{k+1,m+1} \circ \hat{\mu}_{k+1} \circ j_{k+1} \circ \hat{\sigma}_{k+1} \vee 0) \circ \nu^{m,1}_{k+1} \\ &\quad = \iota^{\Omega F_m}_{k+1,m+1} \circ \hat{\mu}_{k+1} \circ j_{k+1} \circ \hat{\sigma}_{k+1}. \end{aligned}$$

From the equation (5.3), we obtain

$$\iota^{\Omega F_m}_{m,m+1}\sigma_m \circ \mu^{m,1}_{k+1} = \iota^{\Omega F_m}_{k+1,m+1} \circ \hat{\mu}_{k+1} \circ j_{k+1} \circ \hat{\sigma}_{k+1}.$$

Therefore we hold the statement by induction.

Finally, we construct a map $\hat{\mu} : \hat{F}_{m+1} \to P^{m+1}\Omega F_m$. Let us consider the exact sequence:

$$[F^{m,1}_m, P^{m+1}\Omega F_m] \xleftarrow{i^{m,1}_m{}^*} [F^{m,1}_{m+1}, P^{m+1}\Omega F_m] \xleftarrow{q^*} [\Sigma K^{m,1}_{m+1}, P^{m+1}\Omega F_m].$$

By the fact that the following diagrams are commutative:

$$\begin{array}{ccccc} F_{m-1} & \xrightarrow{i^F_{m-1,m}} & F_m & \xrightarrow{\sigma_m} & P^m\Omega F_m \\ \downarrow{\scriptstyle \sigma_{m-1}} & & & & \uparrow{\scriptstyle i^{\Omega F_m}_{m-1,m}} \\ P^{m-1}\Omega F_{m-1} & & \xrightarrow{P^{m-1}\Omega i^F_{m-1,m}} & & P^{m-1}\Omega F_m \end{array} \quad \text{and} \quad \begin{array}{ccc} \hat{F}_m & \xrightarrow{\hat{i}_m} & \hat{F}_{m+1} \\ \downarrow{\scriptstyle \hat{\mu}_m} & & \downarrow{\scriptstyle \hat{\mu}_{m+1}} \\ P^m\Omega F_m & \xrightarrow{i^{\Omega F_m}_{m,m+1}} & P^{m+1}\Omega F_m, \end{array}$$

we have

$$\begin{aligned} \hat{\mu}_{m+1} \circ (\sigma_m \times \sigma_1) \circ i^{m,1}_{m+1} &= \hat{\mu}_{m+1} \circ \hat{i}_m \circ \hat{\sigma}_m \\ &= i^{\Omega F_m}_{m,m+1} \circ \hat{\mu}_m \circ \hat{\sigma}_m \end{aligned}$$

and by previous inductive argument ($k = m$ of (5.2)),

$$\begin{aligned} &= \iota^{\Omega F_m}_{m,m+1} \circ \sigma_m \circ \mu^{m,1}_m \\ &= \iota^{\Omega F_m}_{m,m+1} \circ \sigma_m \circ \mu_{m,1} \circ i^{m,1}_{m+1}. \end{aligned}$$

Hence there is a map $\delta_{m+1} : \Sigma K^{m,1}_{m+1} \to P^{m+1}\Omega F_m$ such that

$$\iota^{\Omega F_m}_{m,m+1} \circ \sigma_m \circ \mu_{m,1} = \nabla_{P^{m+1}\Omega F_m} \circ (\hat{\mu}_{m+1} \circ (\sigma_m \times \sigma_1) \vee \delta_{m+1}) \circ \nu^{m,1}_{m+1}. \tag{5.4}$$

To continue calculating, we consider the map $\bar{e} : \hat{E}_{m+1} \to \Sigma K^{m+1}_m$ induced from the bottom left square of the following commutative diagram:

$$\begin{tikzcd} F^{m,1}_m \arrow[r, "i^{m,1}_m"] \arrow[d, "\hat{\sigma}_m"] & F^{m,1}_{m+1} \arrow[r, "q"] \arrow[d, "\sigma_m \times \sigma_1"] & \Sigma K^{m+1}_m \arrow[d, "\Sigma g_m * g_1"] \\ \hat{F}_m \arrow[r, "\hat{i}_m"] \arrow[d, "\hat{e}_m"] & \hat{F}_{m+1} \arrow[r, "\bar{q}"] \arrow[d, "(e^{F_m}_m)^{(\ell)} \times (e^{F_1}_1)^{(\ell)}"] & \hat{E}_{m+1} \arrow[d, dashed, "\bar{e}"] \\ F^{m,1}_m \arrow[r, "i^{m,1}_m"] & F^{m,1}_{m+1} \arrow[r, "q"] & \Sigma K^{m+1}_m, \end{tikzcd}$$

where the map $\hat{e}_m : \hat{F}_m \to F^{m,1}_m$ is $(e^{F_m}_m)^{(\ell)} \times \{*\} \cup (e^{F_{m-1}}_{m-1})^{(\ell)} \times (e^{F_1}_1)^{(\ell)}$. Since $\hat{e}_m \circ \hat{\sigma}_m$ and $(e^{F_m}_m)^{(\ell)} \times (e^{F_1}_1)^{(\ell)} \circ \sigma_m \times \sigma_1$ are homotopic to the identity maps, $\bar{e} \circ \Sigma g_m * g_1$ is homotopic to the identity map of ΣK^{m+1}_m. Then the equation (5.4) is as follows:

$$\begin{aligned} \iota^{\Omega F_m}_{m,m+1} \circ \sigma_m \circ \mu_{m,1} &= \nabla_{P^{m+1}\Omega F_m} \circ (\hat{\mu}_{m+1} \circ (\sigma_m \times \sigma_1) \vee \delta_{m+1}) \circ \nu^{m,1}_{m+1} \\ &= \nabla_{P^{m+1}\Omega F_m} \circ (\hat{\mu}_{m+1} \circ (\sigma_m \times \sigma_1) \vee \delta_{m+1} \circ \bar{e} \circ \Sigma g_m * g_1) \circ \nu^{m,1}_{m+1} \\ &= \nabla_{P^{m+1}\Omega F_m} \circ (\hat{\mu}_{m+1} \vee \delta_{m+1} \circ \bar{e}) \circ ((\sigma_m \times \sigma_1) \vee \Sigma g_m * g_1) \circ \nu^{m,1}_{m+1}. \end{aligned}$$

By Lemma 4.1, we proceed further:

$$= \nabla_{P^{m+1}\Omega F_m} \circ (\hat{\mu}_{m+1} \vee \delta_{m+1} \circ \bar{e}) \circ \hat{\nu}_{m+1} \circ (\sigma_m \times \sigma_1).$$

Therefore we adopt $\nabla_{P^{m+1}\Omega F_m} \circ (\hat{\mu}_{m+1} \vee \delta_{m+1} \circ \bar{e}) \circ \hat{\nu}_{m+1}$ as $\hat{\mu}$. Then we obtain the top square is commutative. And we prove that the bottom square is commutative as follows. By the same argument of the proof of $e^{F_m}_m \circ \delta_k = 0$ for $1 \leq k \leq m$, we have the equation $e^{F_m}_{m+1} \circ \delta_{m+1} = 0$. Thus we obtain

$$\begin{aligned} e^{F_m}_{m+1} \circ \hat{\mu} &= e^{F_m}_{m+1} \circ \nabla_{P^{m+1}\Omega F_m} \circ (\hat{\mu}_{m+1} \vee \delta_{m+1} \circ \bar{e}) \circ \hat{\nu}_{m+1} \\ &= \nabla_{F_m} \circ (e^{F_m}_{m+1} \circ \hat{\mu}_{m+1} \vee e^{F_m}_{m+1} \circ \delta_{m+1} \circ \bar{e}) \circ \hat{\nu}_{m+1} \\ &= \nabla_{F_m} \circ (e^{F_m}_{m+1} \circ \hat{\mu}_{m+1} \vee 0) \circ \hat{\nu}_{m+1} \\ &= e^{F_m}_{m+1} \circ \hat{\mu}_{m+1} \end{aligned}$$

and by the condition (2) of Proposition 2.3 of $\hat{\mu}_{m+1}$, we obtain

$$e^{F_m}_{m+1} \circ \hat{\mu} = \mu_{m,1} \circ \{(e^{F_m}_m)^{(\ell)} \times (e^{F_1}_1)^{(\ell)}\}.$$

$\square$

Thus we have the following commutative diagram:
(5.5)

$$\begin{tikzcd}
F_m & F_m{\times}A \arrow[l, "pr_1"'] \arrow[r, "1\times\alpha"] \arrow[d, "\sigma_m\times\sigma_A"] & F_m{\times}F_1 \arrow[r, "\mu_{m,1}"] \arrow[d, "\sigma_m\times\sigma_1"] & F_m \arrow[d, "\iota^{\Omega F_m}_{m,m+1}\circ\sigma_m"] \\
P^{m+1}\Omega F_m \arrow[from=1-1, d, "\iota^{\Omega F_m}_{m,m+1}\circ\sigma_m"] \arrow[d, "e^{F_m}_{m+1}"] & (P^m\Omega F_m)^{(\ell)}{\times}(\Sigma\Omega A)^{(\ell)} \arrow[l, "\phi"'] \arrow[r, "\chi"] \arrow[d, "(e^{F_m}_m)^{(\ell)}\times(e^A_1)^{(\ell)}"] & \hat{F}_{m+1} \arrow[r, "\hat{\mu}"] \arrow[d, "(e^{F_m}_m)^{(\ell)}\times(e^{F_1}_1)^{(\ell)}"] & P^{m+1}\Omega F_m \arrow[d, "e^{F_m}_{m+1}"] \\
F_m & F_m{\times}A \arrow[l, "pr_1"'] \arrow[r, "1\times\alpha"] & F_m{\times}F_1 \arrow[r, "\mu_{m,1}"] & F_m.
\end{tikzcd}$$

We construct a cone-decomposition of $(P^m\Omega F_m)^{(\ell)}{\times}(\Sigma\Omega A)^{(\ell)}$ of length $m+1$:

$$\{\hat{E}'_k \xrightarrow{\hat{w}'_k} \hat{F}'_{k-1} \xrightarrow{\hat{i}'_{k-1}} \hat{F}'_k \,|\, 1 \le k \le m+1\},$$

by replacing F_1 with A in the construction of the cone-decomposition of $(P^m\Omega F_m)^{(\ell)}{\times}(\Sigma\Omega F_1)^{(\ell)}$. We adopt cofibration sequences

$$\{E^k\Omega F_m \xrightarrow{p^{\Omega F_m}_k} P^{k-1}\Omega F_m \xrightarrow{\iota^{\Omega F_m}_{k-1}} P^k\Omega F_m \,|\, 1 \le k \le m+1\}$$

as a cone-decomposition of $P^{m+1}\Omega F_m$ of length $m+1$. Let D be a homotopy pushout of $(\iota^{\Omega F_m}_{m,m+1})^{(\ell)} \circ pr_1$ and $\hat{\mu} \circ (\mathrm{id}_{(P^m\Omega F_m)^{(\ell)}}{\times}(\Sigma\Omega\alpha)^{(\ell)})$:

$$\begin{tikzcd}
(P^m\Omega F_m)^{(\ell)}{\times}(\Sigma\Omega A)^{(\ell)} \arrow[r, "f^{\rightarrow}"] \arrow[d, "f^{\leftarrow}"] & P^{m+1}\Omega F_m \arrow[d] \\
P^{m+1}\Omega F_m \arrow[r] & D.
\end{tikzcd}$$

Here $f^{\rightarrow} = \hat{\mu} \circ (\mathrm{id}_{(P^m\Omega F_m)^{(\ell)}}{\times}(\Sigma\Omega\alpha)^{(\ell)})$ and $f^{\leftarrow} = (\iota^{\Omega F_m}_{m,m+1})^{(\ell)} \circ pr_1$. We construct a cone-decomposition of D as follows. By the equation $\hat{\mu} \circ \hat{i}_m = \nabla_{P^{m+1}\Omega F_m} \circ (\hat{\mu}_{m+1} \vee \delta_{m+1} \circ \bar{e}) \circ \hat{\nu}_{m+1} \circ \hat{i}_m = \hat{\mu}_{m+1} \circ \hat{i}_m$, we can consider that the restriction of $\hat{\mu}$ on $\hat{F}_k$ is $\hat{\mu}_k$ and $f^{\rightarrow}$ is a filtered map. Since $\hat{E}'_k \xrightarrow{\hat{w}'_k} \hat{F}'_{k-1} \xrightarrow{\hat{i}'_{k-1}} \hat{F}'_k$ is the cofibre sequence, we have

$$\begin{aligned}
e^{F_m}_{k-1} \circ (f^{\rightarrow}|_{\hat{F}'_{k-1}} \circ \hat{w}'_k) &= e^{F_m}_k \circ \iota^{\Omega F_m}_{k-1,k} \circ f^{\rightarrow}|_{\hat{F}'_{k-1}} \circ \hat{w}'_k \\
&= e^{F_m}_k \circ f^{\rightarrow}|_{\hat{F}'_k} \circ \circ \hat{i}'_{k-1} \circ \hat{w}'_k \\
&= e^{F_m}_k \circ f^{\rightarrow}|_{\hat{F}'_k} \circ 0 = 0.
\end{aligned}$$

Using the fibre sequence $E^k\Omega F_m \xrightarrow{p^{\Omega F_m}_k} P^{k-1}\Omega F_m \xrightarrow{e^{F_m}_{k-1}} F_m$, there exists a map $g^{\rightarrow}_k : \hat{E}'_k \to E^k\Omega F_m$ such that the commutativity of the following diagram:

(5.6)
$$\begin{tikzcd}
\hat{E}'_k \arrow[r, "\hat{w}'_k"] \arrow[d, "g^{\rightarrow}_k"] & \hat{F}'_{k-1} \arrow[r, "\hat{i}'_{k-1}"] \arrow[d, "f^{\rightarrow}|_{\hat{F}'_{k-1}}"] & \hat{F}'_k \arrow[d, "f^{\rightarrow}|_{\hat{F}'_k}"] \\
E^k\Omega F_m \arrow[r, "p^{\Omega F_m}_k"] & P^{k-1}\Omega F_m \arrow[r, "\iota^{\Omega F_m}_{k-1,k}"] & P^k\Omega F_m.
\end{tikzcd}$$

Since $f^{\leftarrow}$ is composition of the projection and the inclusion, it is clear that there exists a map $g_k^{\leftarrow} : \hat{E}'_k \to E^k\Omega F_m$ satisfy that the following diagram is commutative:

$$
\begin{tikzcd}
\hat{E}'_k \arrow[r, "\hat{w}'_k"] \arrow[d, "g_k^{\leftarrow}"] & \hat{F}'_{k-1} \arrow[r, "\hat{i}'_{k-1}"] \arrow[d, "f^{\leftarrow}|_{\hat{F}'_{k-1}}"] & \hat{F}'_k \arrow[d, "f^{\leftarrow}|_{\hat{F}'_k}"] \\
E^k\Omega F_m \arrow[r, "p_k^{\Omega F_m}"] & P^{k-1}\Omega F_m \arrow[r, "\iota_{k-1,k}^{\Omega F_m}"] & P^k\Omega F_m.
\end{tikzcd}
\tag{5.7}
$$

Let E_k^P be a homotopy pushout of $g_k^{\rightarrow}$ and $g_k^{\leftarrow}$, and F_k^P be a homotopy pushout of $f^{\rightarrow}|_{\hat{F}'_k}$ and $f^{\leftarrow}|_{\hat{F}'_k}$, then using diagrams (5.6) and (5.7) and using the universal property of the homotopy pushout, we have the following diagram such that the front column $E_k^P \to F_{k-1}^P \to F_k^P$ is a cofibration:

$$
\begin{tikzcd}
 & & \hat{E}'_k \arrow[dll, "g_k^{\leftarrow}"'] \arrow[drr, "g_k^{\rightarrow}"] \arrow[d, "\hat{w}'_k"] & & \\
E^k\Omega F_m \arrow[d, "p_k^{\Omega F_m}"] \arrow[drrr] & & \hat{F}'_{k-1} \arrow[dll, "f^{\leftarrow}|_{\hat{F}'_{k-1}}"'] \arrow[drr, "f^{\rightarrow}|_{\hat{F}'_{k-1}}"] \arrow[d, "\hat{i}'_{k-1}"] & & E^k\Omega F_m \arrow[d, "p_k^{\Omega F_m}"] \arrow[dl] \\
P^{k-1}\Omega F_m \arrow[d, "\iota_{k-1,k}^{\Omega F_m}"] \arrow[drrr] & & \hat{F}'_k \arrow[dll, "f^{\leftarrow}|_{\hat{F}'_k}"'] \arrow[drr, "f^{\rightarrow}|_{\hat{F}'_k}"] & E_k^P \arrow[d] & P^{k-1}\Omega F_m \arrow[d, "\iota_{k-1,k}^{\Omega F_m}"] \arrow[dl] \\
P^k\Omega F_m \arrow[drrr] & & & F_{k-1}^P \arrow[d] & P^{k-1}\Omega F_m \arrow[dl] \\
 & & & F_k^P. &
\end{tikzcd}
$$

Thus we obtain a cone-decomposition of D of length $m+1$:

$$\{E_k^P \to F_{k-1}^P \to F_k^P \mid 1 \le k \le m+1\}.$$

Therefore we have the inequalities

$$\mathrm{cat}(D) \le \mathrm{Cat}(D) \le m+1.$$

Recall the horizontal top and bottom lines of the diagram (5.5). The homotopy pushout of these lines are $G \cup_\psi G{\times}CA$. Since dimensions of F_m, F_1 and A are less than or equal to l, all composition of columns in the diagram (5.5) are homotopic to identity maps. By the universal property of the homotopy pushout, we obtain a composite map $D \to G \cup_\psi G{\times}CA \simeq E \to D$ which is homotopic to the identity map. Thus D dominates E and we have

$$\mathrm{cat}(E) \le \mathrm{cat}(D) \le \mathrm{Cat}(D) \le m+1.$$

6. Application of Theorem 1.4

We want to determine the L-S category of SO(10) by applying the principal bundle $p : \mathrm{SO}(10) \to S^9$ to Theorem 1.4. First, we estimate the lower bound of cat(SO(10)). For the field k of characteristic 2, the ring structure of the cohomology of SO(10) is

$$H^*(\mathrm{SO}(10); k) \cong P_k[x_1, x_3, x_5, x_7, x_9]/(x_1^{16}, x_3^4, x_5^2, x_7^2, x_9^2),$$

where $\deg x_i = 1$. Hence, we have

$$21 \leq \mathrm{cup}(\mathrm{SO}(10); k) \leq \mathrm{cat}(\mathrm{SO}(10)).$$

Next, we estimate the upper bound by using Theorem 1.4. We consider the cone-decomposition of SO(9). The cone-decomposition of Spin(7) is given by Iwase, Mimura and Nishimoto[7]. We denote this cone-decomposition by the following:

$$* \subset F_1' = \Sigma\mathbb{C}\mathrm{P}^3 \subset F_2' \subset F_3' \subset F_4' \subset F_5' \simeq \mathrm{Spin}(7).$$

By Iwase, Mimura and Nishimoto [8], we can write the cone-decomposition of length 20

$$\{K_i \to F_{i-1} \to F_i \mid 1 \leq i \leq 20,\ F_0 = \{*\} \text{ and } F_{20} = \mathrm{SO}(9)\}$$

by using the filtration F_i' and principal bundle $\mathrm{Spin}(7) \hookrightarrow \mathrm{SO}(9) \to \mathbb{R}\mathrm{P}^{15}$. We find that the first filter F_1 is the space $\Sigma\mathbb{C}\mathrm{P}^3 \vee \mathrm{S}^1$. We consider the bundle $p : \mathrm{SO}(10) \to S^9$ and $p' : \mathrm{SU}(5) \to S^9$, and the following diagram:

$$\begin{tikzcd}
\Sigma\mathbb{C}\mathrm{P}^3 \arrow[r, hook] & \mathrm{SU}(4) \arrow[r, hook] \arrow[d, hook] & \mathrm{SO}(9) \arrow[d, hook] \\
 & \mathrm{SU}(5) \arrow[r, hook] \arrow[dr, "p'"] & \mathrm{SO}(10) \arrow[d, "p"] \\
 & S^8 \arrow[r, hook] \arrow[uur, "\alpha"] \arrow[uu, dashed] \arrow[uul, dashed] & S^9.
\end{tikzcd}$$

Here $\alpha : S^8 \to \mathrm{SO}(9)$ is a characteristic map of the bundle $p : \mathrm{SO}(10) \to S^9$. By Steenrod [13], is homotopic to the characteristic map $\alpha' : S^8 \to \mathrm{SU}(4)$ in SO(9). Also, by Yokota [15], the suspension of the covering map $\Sigma\gamma_3 : S^8 \to \Sigma\mathbb{C}\mathrm{P}^3$ which provide a cellular decomposition of the complex projective space correspond with the characteristic map α'. Therefore the characteristic map α is compressible into $\Sigma\mathbb{C}\mathrm{P}^3 \subset F_1$ and $H_1(\alpha) = 0 \in \pi_8(\Omega\Sigma\mathbb{C}\mathrm{P}^3 * \Omega\Sigma\mathbb{C}\mathrm{P}^3)$. Hence we obtain

Theorem 6.1. $\mathrm{cat}(\mathrm{SO}(10)) = 21$.

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